

NEGATIVE VALUES OF THE RIEMANN ZETA FUNCTION ON THE CRITICAL LINE

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ABSTRACT. We investigate the intersections of the curve $\mathbb{R} \ni t \mapsto \zeta(\frac{1}{2} + it)$ with the real axis. We show unconditionally that the zeta-function takes arbitrarily large positive and negative values on the critical line.

KEYWORDS: Riemann zeta-function, value-distribution, critical line

MATHEMATICAL SUBJECT CLASSIFICATION: 11M06

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

We investigate the value-distribution of the Riemann zeta-function $\zeta(s)$ on the critical line $s = \frac{1}{2} + i\mathbb{R}$. Recently Soundararajan [11] succeeded to prove

$$(1) \quad \max_{t \in [T, 2T]} |\zeta(\frac{1}{2} + it)| \gg \exp \left((1 + o(1)) \sqrt{\frac{\log T}{\log \log T}} \right) \quad \text{as } T \rightarrow \infty.$$

This result as well as all other Ω -estimates for the Riemann zeta function does not provide special information about the location of the large values of the Riemann zeta function in the complex plane. For this purpose we study the set of intersection points of the curve $t \mapsto \zeta(\frac{1}{2} + it)$ with straight lines $e^{i\phi}\mathbb{R}$ through the origin and prove upper and lower bounds for associated discrete moments. In particular, we prove that $\zeta(\frac{1}{2} + it)$ takes arbitrarily large positive values as well as arbitrarily large negative values.

Recall the functional equation for the zeta-function,

$$(2) \quad \zeta(s) = \Delta(s)\zeta(1-s), \quad \text{where } \Delta(s) := 2^s \pi^{s-1} \Gamma(1-s) \sin(\frac{\pi s}{2}).$$

It follows immediately that $\Delta(s)\Delta(1-s) = 1$, hence $\Delta(\frac{1}{2} + it)$ lies on the unit circle for real t . Given an angle $\phi \in [0, \pi)$, denote by $t_n(\phi)$ with $n \in \mathbb{N}$ the positive roots of the equation

$$e^{2i\phi} = \Delta(\frac{1}{2} + it)$$

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in ascending order. These roots correspond to intersections of the curve $t \mapsto \zeta(\frac{1}{2} + it)$ with straight lines $e^{i\phi}\mathbb{R}$ through the origin (see Kalpokas and Steuding [7]). Of special interest are intersections with the real line; in this case $\phi = 0$ and the roots are called Gram points (after Gram [5] who observed that the first of those roots separate consecutive zeta zeros on the critical line).

Building on work of Rudnick and Soundararajan [10], resp. Milinovich and Ng [8] we shall establish a lower bound of the expected order for those discrete moments with arbitrary rational exponents:

Theorem 1. *For any rational $k \geq 1$ and any $\phi \in [0, \pi)$, as $T \rightarrow \infty$,*

$$\sum_{0 < t_n(\phi) \leq T} \left| \zeta\left(\frac{1}{2} + it_n(\phi)\right) \right|^{2k} \gg T(\log T)^{k^2+1}.$$

Continuing recent work of Kalpokas and Steuding [7], we shall derive an asymptotic formula for the third discrete moment:

Theorem 2. *For any $\phi \in [0, \pi)$, as $T \rightarrow \infty$,*

$$\begin{aligned} & \sum_{0 < t_n^\phi \leq T} \zeta\left(\frac{1}{2} + it_n(\phi)\right)^3 \\ &= 2e^{3i\phi}(\cos \phi) \frac{T}{2\pi} P_3\left(\log \frac{T}{2\pi}\right) + 2e^{3i\phi}(\cos 3\phi) \frac{T}{2\pi} \log \frac{T}{2\pi e} + O\left(T^{\frac{1}{2}+\varepsilon}\right), \end{aligned}$$

where P_3 is a computable polynomial of degree three.

Combining the above theorems we shall deduce that there are arbitrary large positive and negative values of the Riemann zeta-function on the critical line. More generally, we shall show that the corresponding statement holds with respect to any direction ϕ . Recall that $e^{-i\phi}\zeta(\frac{1}{2} + it_n(\phi))$ is real. Hence, we may write $t_n^+(\phi)$ in place of $t_n(\phi)$ if $e^{-i\phi}\zeta(\frac{1}{2} + it_n(\phi)) \geq 0$ and $t_n^-(\phi)$ if $e^{-i\phi}\zeta(\frac{1}{2} + it_n(\phi)) < 0$.

Corollary 3. *For any $\phi \in [0, \pi)$, there are arbitrary large positive and negative values of $e^{-i\phi}\zeta(\frac{1}{2} + it_n(\phi))$. More precisely,*

$$\max_{0 < t_n^\pm(\phi) \leq T} \left| \zeta\left(\frac{1}{2} + it_n^\pm(\phi)\right) \right| \gg (\log T)^{\frac{5}{4}}.$$

If Riemann hypothesis is assumed for any arbitrary small $\delta > 0$ we have

$$\max_{0 < t_n^\pm(\phi) \leq T} \left| \zeta\left(\frac{1}{2} + it_n^\pm(\phi)\right) \right| \gg (\log T)^{\frac{3}{2}-\delta}.$$

If we are interested only in extreme values on a given straight line without information on which of the two half-lines the value is taken, we can obtain estimates comparable to (1) due to Soundararajan [11] except for the imaginary axis:

Corollary 4. *Let $\phi \neq \frac{\pi}{2}$ and $\phi \in [0, \pi)$, then*

$$\max_{0 < t_n(\phi) \leq T} |\zeta(\frac{1}{2} + it_n(\phi))| \gg \exp\left(\left(\frac{1}{2} + o(1)\right) \sqrt{\frac{\log T}{\log \log T}}\right).$$

The paper is organized as follows. In the next section we provide some preliminary results. In Section 3 we prove the key Proposition 9 which leads to both, Theorem 2 and Corollary 4. Section 4 contains the proof of Theorem 1 and Corollary 3.

2. PRELIMINARIES

Recall the function $\Delta(s)$ defined in (2). It is well-known that

$$(3) \quad \Delta(\sigma + it) = \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2} - \sigma - it} \exp(i(t + \frac{\pi}{4}))(1 + O(|t|^{-1})) \quad \text{for } |t| \geq 1$$

uniformly for any σ from a bounded interval. Hence,

$$(4) \quad \frac{1}{\Delta(s) - e^{2i\phi}} = \frac{-e^{-2i\phi}}{1 - e^{-2i\phi}\Delta(s)} = -e^{-2i\phi} \left(1 + \sum_{k=1}^{\infty} e^{-2ki\phi} \Delta(s)^k\right)$$

for $\sigma > \frac{1}{2}$. Obviously, $\Delta(\frac{1}{2} + it)$ is a complex number from the unit circle for $t \in \mathbb{R}$. Moreover, $\Delta'(\frac{1}{2} + it)$ is non-vansishing for sufficiently large t as follows from the asymptotic formula

$$(5) \quad \frac{\Delta'}{\Delta}(\sigma + it) = -\log \frac{|t|}{2\pi} + O(|t|^{-1}) \quad \text{for } |t| \geq 1.$$

Next we introduce certain Dirichlet polynomials

$$(6) \quad X(s) = \sum_{n \leq X} \frac{x_n}{n^s}, \quad Y(s) = \sum_{m \leq Y} \frac{y_m}{m^s},$$

where $X, Y \leq T$. Moreover, we define the following quantities

$$\mathcal{X}_0 = \max_{n \leq X} |x_n|, \quad \mathcal{Y}_0 = \max_{m \leq Y} |y_m|, \quad \mathcal{X}_1 = \sum_{n \leq X} \frac{|x_n|}{n}, \quad \mathcal{Y}_1 = \sum_{m \leq Y} \frac{|y_m|}{m}.$$

and we set

$$X_1(s) = \sum_{n \leq X} \frac{\bar{x}_n}{n^s}, \quad Y_1(s) = \sum_{m \leq Y} \frac{\bar{y}_m}{m^s}.$$

We shall use a variation of Lemma 5.1 from Ng [9]:

Lemma 5. *Suppose the series $f(s) = \sum_{n=1}^{\infty} \alpha_n n^{-s}$ converges absolutely for $\operatorname{Re} s > 1$ and $\sum_{n=1}^{\infty} |\alpha_n| n^{-\sigma} \ll (\sigma - 1)^{-\gamma}$ for some $\gamma \geq 0$ as $\sigma \rightarrow 1 + 0$. Next, let $X(s)$ and*

$Y(s)$ be Dirichlet polynomials as defined in (6). Then, uniformly for $a \in (1, 2]$,

$$\begin{aligned} J &= \frac{1}{2\pi i} \int_{a-i}^{a+iT} f(s) X(s) Y(1-s) \frac{\Delta'}{\Delta}(s) ds \\ &= -\frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) \sum_{\substack{m \leq X \\ mn \leq Y}} \frac{\alpha_n x_m y_{mn}}{mn} + O \left(\frac{Y^a (\log T)^2 \mathcal{X}_0 \mathcal{Y}_0}{(a-1)^{\gamma+1}} \right), \end{aligned}$$

where the implicit constant is absolute.

Proof. Changing the order of summation and the integration, we get

$$J = \sum_{n=1}^{\infty} \sum_{m \leq X, k \leq Y} \left(\frac{k}{mn} \right)^a \frac{\alpha_n x_m y_k}{k} \frac{1}{2\pi} \int_1^T \frac{\Delta'}{\Delta}(a+it) \left(\frac{k}{mn} \right)^{it} dt.$$

Next, the contribution of O - term from (5) to J does not exceed in order

$$\mathcal{X}_0 \mathcal{Y}_0 (\log T) \sum_{n=1}^{\infty} \frac{|\alpha_n|}{n^a} \sum_{m \leq X} \frac{1}{m^a} \sum_{k \leq Y} k^{a-1} \ll \frac{Y^a (\log T)^2 \mathcal{X}_0 \mathcal{Y}_0}{(a-1)^{\gamma}}.$$

Extracting the diagonal term (when $k = mn$) in the above expression for J , we get

$$J = \left(- \int_1^T \log \frac{t}{2\pi} \frac{dt}{2\pi} \right) \sum_{\substack{m \leq X \\ mn \leq Y}} \frac{\alpha_n x_m y_{mn}}{mn} + O(J_1) + O \left(\frac{Y^a (\log T)^2 \mathcal{X}_0 \mathcal{Y}_0}{(a-1)^{\gamma}} \right),$$

where

$$\begin{aligned} J_1 &= \sum_{n=1}^{\infty} \sum_{\substack{m \leq X, k \leq Y \\ mn \neq k}} \left(\frac{k}{mn} \right)^a \frac{|\alpha_n x_m y_k|}{k} |j_{k,mn}| \\ &\leq Y^{a-1} \mathcal{X}_0 \mathcal{Y}_0 \sum_{n=1}^{\infty} \sum_{\substack{m \leq X, k \leq Y \\ mn \neq k}} \frac{|\alpha_n|}{(mn)^a} |j_{k,mn}| \end{aligned}$$

with

$$j_{k,r} = \int_1^T \left(\log \frac{t}{2\pi} \right) \left(\frac{k}{r} \right)^{it} \frac{dt}{2\pi}.$$

Integrating by parts shows for $k \neq r$ that

$$|j_{k,r}| = \left| \int_1^T \frac{\log(t/(2\pi))}{2\pi \log(k/r)} d \left(\frac{k}{r} \right)^{it} \right| \leq \frac{2}{\pi} \log \frac{T}{2\pi} \left| \log \frac{k}{r} \right|^{-1}.$$

Setting $r = mn$, $\beta_r = \sum_{n|r} |\alpha_n|$ in the expression for J_1 , we have

$$J_1 \ll Y^{a-1} (\log T) \mathcal{X}_0 \mathcal{Y}_0 \sum_{k \leq Y} \sum_{r=1}^{\infty} \frac{\beta_r}{r^a} \left| \log \frac{k}{r} \right|^{-1}.$$

Recall that $\zeta(s)$ has a simple pole at $s = 1$. Thus, the contribution of the terms with $r \leq k/2$ and $r > 3k/2$ does not exceed in order

$$Y^{a-1} (\log T) \mathcal{X}_0 \mathcal{Y}_0 \cdot Y \sum_{r=1}^{\infty} \frac{\beta_r}{r^a} \ll Y^a \mathcal{X}_0 \mathcal{Y}_0 (\log T) \sum_{n=1}^{\infty} \frac{|\alpha_n|}{n^a} \sum_{m=1}^{\infty} \frac{1}{m^a} \ll \frac{Y^a (\log T) \mathcal{X}_0 \mathcal{Y}_0}{(a-1)^{\gamma+1}}.$$

For $k/2 < r \leq 3k/2$, $r \neq k$ we set $r = k + \nu$; since $|\log(k/r)|^{-1} \ll k|\nu|^{-1}$, the corresponding part of J_1 can be estimated as follows:

$$\begin{aligned} & Y^{a-1}(\log T) \mathcal{X}_0 \mathcal{Y}_0 \sum_{k \leq Y} \sum_{0 < |\nu| \leq k/2} \frac{\beta_{k+\nu}}{(k+\nu)^a} \frac{k}{|\nu|} \\ & \ll Y^a(\log T) \mathcal{X}_0 \mathcal{Y}_0 \sum_{0 < |\nu| \leq Y/2} \frac{1}{|\nu|} \sum_{2|\nu| \leq k \leq Y} \frac{\beta_{k+\nu}}{(k+\nu)^a} \\ & \ll Y^a(\log T)^2 \mathcal{X}_0 \mathcal{Y}_0 \sum_{k=1}^{\infty} \frac{\beta_k}{k^a} \ll \frac{Y^a(\log T)^2 \mathcal{X}_0 \mathcal{Y}_0}{(a-1)^{\gamma+1}}. \end{aligned}$$

The lemma is proved. \square

Our next lemma is a variation of Gonek's lemma:

Lemma 6. *Suppose the series $f(s) = \sum_{n=1}^{\infty} \alpha_n n^{-s}$ converges absolutely in the half-plane $\operatorname{Re} s > 1$, $\sum_{n=1}^{\infty} |\alpha_n| n^{-\sigma} \ll (\sigma - 1)^{-\gamma}$ for some $\gamma \geq 0$ as $\sigma \rightarrow 1 + 0$ and $\alpha_n \ll n^\varepsilon$ for any $\varepsilon > 0$. Then we have, for any fixed integer $m \geq 0$ and $c \geq 1$ uniformly for $a \in (1, 2]$,*

$$\begin{aligned} & \frac{1}{2\pi i} \int_{a+ic}^{a+iT} f(s) \Delta(1-s) \left(\frac{\Delta'}{\Delta}(s) \right)^m ds \\ & = (-1)^m \sum_{n \leq \frac{T}{2\pi}} \alpha_n (\log n)^m + O\left(\frac{T^{a-\frac{1}{2}}}{(a-1)^\gamma} (\log T)^m + T^{\frac{1}{2}+\varepsilon} \right). \end{aligned}$$

For the proof we refer to Lemma 5 from [4] (in the original paper the remainder term is not uniform in $a > 1$).

Finally, we consider the divisor function $r \mapsto d(r) := \sum_{d|r} 1$ and some of its generalizations, respectively. For any positive real κ and $\operatorname{Re} s > 1$ define the numbers $d_\kappa(n)$ by

$$\zeta(s)^\kappa = \sum_{n=1}^{\infty} d_\kappa(n) n^{-s}.$$

Notice that $n \mapsto d_\kappa(n)$ is a multiplicative function, on a prime power p^j given by

$$d_\kappa(p^j) = \frac{\Gamma(\kappa + j)}{\Gamma(\kappa)j!}$$

(see [3, formula (5)]). We continue with two lemmas on this generalized divisor function:

Lemma 7. *Let κ be a positive real number and n a positive integer.*

- (1) *For $\kappa \geq 0$ and $n \geq 1$ we have $d_\kappa(n) \geq 0$.*
- (2) *For fixed n , $d_\kappa(n)$ increases with respect to κ .*
- (3) *For fixed $\kappa \geq 0$ and $\varepsilon > 0$ we have $d_\kappa(n) \ll n^\varepsilon$.*

(4) If j is an integer then

$$d_{\kappa j}(n) = \sum_{n=n_1 n_2 \dots n_j} d_{\kappa}(n_1) d_{\kappa}(n_2) \dots d_{\kappa}(n_j).$$

For a proof we refer to [3].

Lemma 8. For any fixed positive real λ, μ we have, as $x \rightarrow \infty$,

$$\sum_{n \leq x} d_{\lambda}(n) d_{\mu}(n) n^{-1} \asymp_{\lambda, \mu} (\log x)^{\lambda \mu},$$

and, if $\lambda \geq 2$ is a positive integer,

$$\sum_{n \leq x} d_{\lambda}(n) = x P_{\lambda-1}(\log x) + O(x^{\delta_{\lambda}}),$$

where $P_{\lambda-1}$ is a computable polynomial of degree $\lambda-1$ and δ_{λ} is a positive quantity strictly less than one.

These assertions can be established by the standard technique based on Perron's formula and contour integration (see [6], Chapter 13).

3. THE KEY PROPOSITION

In order to prove Theorem 1 and Corollary 4 we consider the discrete moments

$$(7) \quad S_1(T) = \sum_{0 < t_n(\phi) \leq T} \zeta\left(\frac{1}{2} - it_n(\phi)\right) X\left(\frac{1}{2} + it_n(\phi)\right) Y\left(\frac{1}{2} - it_n(\phi)\right)$$

and

$$(8) \quad S_2(T) = \sum_{0 < t_n(\phi) \leq T} |X\left(\frac{1}{2} + it_n(\phi)\right)|^2.$$

with Dirichlet polynomials $X(s)$ and $Y(s)$ which will be specified later. Our first aim is the following

Proposition 9. Let $X(s)$ and $Y(s)$ be Dirichlet polynomials as defined in (6). Then for any $\phi \in [0, \pi)$, as $T \rightarrow \infty$,

$$(9) \quad S_1(T) = \frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) \left(e^{-2i\phi} \sum_{\substack{m \leq X \\ mn \leq Y}} \frac{x_m y_{mn}}{mn} + \sum_{\substack{m \leq Y \\ mn \leq X}} \frac{y_m x_{mn}}{mn} \right) + O(R_1),$$

where

$$R_1 = (X + Y) \sqrt{T} (\log T)^2 \mathcal{X}_1 \mathcal{Y}_1 + (X + Y) (\log T)^4 \mathcal{X}_0 \mathcal{Y}_0;$$

moreover,

$$(10) \quad S_2(T) = \frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) \sum_{n \leq X} \frac{|x_n|^2}{n} + O(R_2),$$

where

$$R_2 = X\sqrt{T}(\log T)^2 \sum_{n \leq X} \frac{|x_n|^2}{n} + X(\log T)^3 \mathcal{X}_0^2.$$

All implicit constants are absolute.

3.1. Proof of Proposition 9.

3.1.1. *Proof of (9).* We begin with the estimations

$$\begin{aligned} |\zeta(\tfrac{1}{2} + it)| &\ll T^{1/6}(\log T)^{3/2}, \\ |X(\tfrac{1}{2} + it)| &\leq \sum_{n \leq X} \frac{|x_n|}{\sqrt{n}} = \sum_{n \leq X} \sqrt{n} \frac{|x_n|}{n} \leq \sqrt{X} \mathcal{X}_1, \quad |Y(\tfrac{1}{2} + it)| \leq \sqrt{Y} \mathcal{Y}_1; \end{aligned}$$

the first one is a well-known bound from zeta-function theory (see [6]), whereas the second and third one are straightforward. Hence, it is sufficient to obtain (9) for the sum over the segment $c < t_n(\phi) \leq T$, where $c > 32\pi$ is a large absolute constant (32π comes from the inequality $2(\frac{t}{2\pi})^{-\frac{1}{2}} < \frac{1}{2}$ that is used in the proof).

Next, without loss of generality we may set $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$. Indeed, otherwise we may replace T by the closest value T_1 of such type. Then the error of such replacement in the right-hand side of (9) is bounded by

$$\begin{aligned} &\left(\log \frac{T}{2\pi}\right)^{-1} \left(\log \frac{T}{2\pi}\right) \left(\sum_{\substack{m \leq X \\ mn \leq Y}} \frac{|x_m y_{mn}|}{mn} + \sum_{\substack{m \leq Y \\ mn \leq X}} \frac{|y_m x_{mn}|}{mn} \right) \\ &\ll \mathcal{X}_0 \mathcal{Y}_0 \sum_{mn \leq T} \frac{1}{mn} \ll \mathcal{X}_0 \mathcal{Y}_0 \sum_{r \leq T} \frac{d(r)}{r} \ll \mathcal{X}_0 \mathcal{Y}_0 (\log T)^2, \end{aligned}$$

where we have used Lemma 8 and the asymptotics $T \log T$ for the number of $t_n(\phi) \leq T$ (see Theorem 1 from [7]). Since the points $s = \frac{1}{2} + it_n(\phi)$ are the roots of the function $\Delta(s) - e^{2i\phi}$, the sum in question can be rewritten as a contour integral:

$$\begin{aligned} &\sum_{c < t_n(\phi) \leq T} \zeta(\tfrac{1}{2} - it_n(\phi)) X(\tfrac{1}{2} + it_n(\phi)) Y(\tfrac{1}{2} - it_n(\phi)) \\ &= \frac{1}{2\pi i} \int_{\square} \zeta(1-s) X(s) Y(1-s) \frac{\Delta'(s)}{\Delta(s) - e^{2i\phi}} ds; \end{aligned}$$

here \square stands for the counterclockwise oriented rectangular contour with vertices $a + ic$, $a + iT$, $1 - a + iT$, $1 - a + ic$, where $a = 1 + (\log T)^{-1}$. Let \mathcal{I}_1 and \mathcal{I}_3 be integrals over right and left sides of contour, and \mathcal{I}_2 and \mathcal{I}_4 be the integrals over the top and bottom edges of the contour. We may assume the constant c so large that the relations

$$|\Delta(a + it)| = \left(\frac{t}{2\pi}\right)^{1/2-a} (1 + O(t^{-1})) \leq 2 \left(\frac{t}{2\pi}\right)^{-1/2} < \frac{1}{2}$$

hold for any $t > c$. In view of (4) we have

$$\begin{aligned}\mathcal{I}_1 &= \frac{1}{2\pi i} \int_{a+ic}^{a+iT} \zeta(1-s)X(s)Y(1-s) \frac{\Delta'(s)}{\Delta(s) - e^{2i\phi}} ds \\ &= -\frac{e^{-2i\phi}}{2\pi i} \int_{a+ic}^{a+iT} \frac{\zeta(s)}{\Delta(s)} X(s)Y(1-s)\Delta'(s) \left(1 + \sum_{k=1}^{\infty} e^{-2ik\phi} \Delta(s)^k\right) ds \\ &= -e^{-2i\phi}(j_1 + j_2),\end{aligned}$$

where

$$\begin{aligned}j_1 &= \frac{1}{2\pi i} \int_{a+ic}^{a+iT} \zeta(s)X(s)Y(1-s) \frac{\Delta'}{\Delta}(s) ds, \\ j_2 &= \frac{1}{2\pi i} \int_{a+ic}^{a+iT} \zeta(s)X(s)Y(1-s) \frac{\Delta'}{\Delta}(s) \sum_{k=1}^{\infty} e^{-2ik\phi} \Delta(s)^k ds.\end{aligned}$$

We observe for $s = a + it$

$$\begin{aligned}|X(a+it)| &\leq \sum_{n \leq X} \frac{|x_n|}{n^a} \leq \mathcal{X}_1, \quad |Y(1-a-it)| \leq \sum_{m \leq Y} \frac{m^a |y_m|}{m} \ll Y\mathcal{Y}_1, \\ \left| \sum_{k=1}^{\infty} e^{-2ik\phi} \Delta(a+it)^k \right| &\leq 2 \left(\frac{t}{2\pi} \right)^{-1/2} \sum_{k=0}^{\infty} \frac{1}{2^k} \ll t^{-1/2}.\end{aligned}$$

Thus, we have

$$|j_2| \ll \zeta(a)Y\mathcal{X}_1\mathcal{Y}_1 \int_c^T \frac{\log t \, dt}{\sqrt{t}} \ll Y\sqrt{T}(\log T)^2 \mathcal{X}_1\mathcal{Y}_1.$$

Applying Lemma 5 to j_1 we get

$$\mathcal{I}_1 = e^{-2i\phi} \frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) \sum_{\substack{m \leq X \\ mn \leq Y}} \frac{x_m y_{mn}}{mn} + O(Y\sqrt{T}(\log T)^2 \mathcal{X}_1\mathcal{Y}_1 + Y(\log T)^4 \mathcal{X}_0\mathcal{Y}_0).$$

In a similar way we may compute \mathcal{I}_3 . We observe

$$\mathcal{I}_3 = -\frac{1}{2\pi} \int_c^T \zeta(a-it)X(1-a+it)Y(a-it) \frac{\Delta'(1-a+it)}{\Delta(1-a+it) - e^{2i\phi}} dt.$$

This in combination with $\overline{X}(s) = X_1(\overline{s})$, $\overline{Y}(s) = Y_1(\overline{s})$ yields

$$\overline{\mathcal{I}}_3 = -\frac{1}{2\pi i} \int_{a+ic}^{a+iT} \zeta(s)X_1(1-s)Y_1(s) \frac{\Delta'(1-s)}{\Delta(1-s) - e^{-2i\phi}} ds.$$

In view of (4) we find $\overline{\mathcal{I}}_3 = -(j_3 + j_4)$, where the expressions for j_3 and j_4 can be obtained by the replacing $X(s)$ with $Y_1(s)$ and $Y(1-s)$ with $X_1(1-s)$ in the expressions for j_1 and j_2 . Applying Lemma 4 to j_3 and estimating j_4 similarly to j_2 , we get

$$\overline{\mathcal{I}}_3 = \frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) \sum_{\substack{m \leq Y \\ mn \leq X}} \frac{\overline{y}_m \overline{x}_{mn}}{mn} + O(X\sqrt{T}(\log T)^2 \mathcal{X}_1\mathcal{Y}_1 + X(\log T)^4 \mathcal{X}_0\mathcal{Y}_0).$$

In order to estimate \mathcal{I}_2 we first note that the following inequalities hold along the line segment of integration:

$$|\zeta(1-s)| \ll \sqrt{T} \log T, \quad |X(s)| \leq \sum_{n \leq X} \frac{|x_n|}{n} n^{1-\sigma} \ll X^{1-\sigma} \mathcal{X}_1,$$

$$|Y(1-s)| \leq \sum_{n \leq Y} \frac{|y_n|}{n} n^\sigma \ll Y^\sigma \mathcal{Y}_1,$$

and, finally,

$$\begin{aligned} |\zeta(1-s)X(s)Y(1-s)| &\ll \sqrt{T} \mathcal{X}_1 \mathcal{Y}_1 X \left(\frac{Y}{X} \right)^\sigma \log T \\ &\ll X \sqrt{T} \mathcal{X}_1 \mathcal{Y}_1 \left\{ \left(\frac{Y}{X} \right)^{1-a} + \left(\frac{Y}{X} \right)^a \right\} \log T \\ &\ll (X+Y) \sqrt{T} (\log T) \mathcal{X}_1 \mathcal{Y}_1. \end{aligned}$$

Next, by (5) we get

$$\frac{\Delta'(s)}{\Delta(s) - e^{2i\phi}} = \frac{\Delta'(s)}{\Delta(s)} \left(1 + \frac{e^{2i\phi}}{\Delta(s) - e^{2i\phi}} \right) \ll (\log T) \left(1 + \frac{1}{|\Delta(s) - e^{2i\phi}|} \right).$$

The second term in the brackets is bounded by an absolute constant. Indeed, in the case $\sigma \geq \frac{1}{2} + \frac{1}{3} \left(\log \frac{T}{2\pi} \right)^{-1}$ by (3) we have

$$|\Delta(\sigma + iT)| = \left(\frac{T}{2\pi} \right)^{1/2-\sigma} (1 + O(T^{-1})) \leq e^{-1/3} (1 + O(T^{-1})) < \frac{1}{2},$$

and hence $|\Delta(s) - e^{2i\phi}| \geq 1 - |\Delta(s)| > \frac{1}{2}$. Similarly, in the case $\sigma \leq \frac{1}{2} - \frac{1}{3} \left(\log \frac{T}{2\pi} \right)^{-1}$ we have

$$|\Delta(\sigma + iT)| \geq e^{1/3} (1 + O(T^{-1})) > \frac{4}{3}, \quad |\Delta(s) - e^{2i\phi}| > \frac{4}{3} - 1 = \frac{1}{3}.$$

Finally, let

$$\frac{1}{2} - \frac{1}{3} \left(\log \frac{T}{2\pi} \right)^{-1} < \sigma < \frac{1}{2} + \frac{1}{3} \left(\log \frac{T}{2\pi} \right)^{-1}.$$

Then, using the relations

$$\Delta\left(\frac{1}{2} + iT\right) = e^{-2i\vartheta(T)}, \quad \Delta(\sigma + iT) = \tau e^{-2i\vartheta(T)} (1 + O(T^{-1})),$$

where $\tau = (T/(2\pi))^{1/2-\sigma}$ and $\vartheta = \vartheta(T)$ denotes the increment of any fixed continuous branch of the argument of $\pi^{-s/2} \Gamma(s/2)$ along the line segment with end-points $s = \frac{1}{2}$ and $s = \frac{1}{2} + iT$ (check [2]), we have $e^{-1/3} \leq \tau \leq e^{1/3}$ and

$$\begin{aligned} \Delta(\sigma + iT) - e^{2i\phi} &= \left(\Delta(\sigma + iT) - \Delta\left(\frac{1}{2} + iT\right) \right) + \left(\Delta\left(\frac{1}{2} + iT\right) - e^{2i\phi} \right) \\ &= (\tau - 1) e^{-2i\vartheta} - 2ie^{i(\phi-\vartheta)} \sin(\phi + \vartheta) + O(T^{-1}) \\ &= e^{-i\vartheta} \left((\tau - 1) \cos \vartheta + 2 \sin(\vartheta + \phi) \sin \phi - \right. \\ &\quad \left. - i((\tau - 1) \sin \vartheta + 2 \sin(\vartheta + \phi) \cos \phi) \right) + O(T^{-1}). \end{aligned}$$

Thus we obtain

$$\begin{aligned} |\Delta(\sigma + iT) - e^{2i\phi}|^2 &= (\tau - 1)^2 + 4\tau \sin^2(\vartheta + \phi) + O(T^{-1}) \\ &\geq 4\tau \sin^2(\vartheta + \phi) + O(T^{-1}). \end{aligned}$$

Using the fact that $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$ for some ν , we finally get

$$\sin^2(\vartheta + \phi) = \sin^2\left(\pi\nu + \frac{\pi}{2} + O(T^{-1})\right) \geq \sin^2 \frac{\pi}{3} = \frac{3}{4}$$

and hence, for sufficiently large T ,

$$|\Delta(\sigma + iT) - e^{2i\phi}|^2 \geq 4 \cdot \frac{3}{4} e^{-1/3} + O(T^{-1}) > 2.$$

Thus, $|\Delta(s) - e^{2i\phi}| > \frac{1}{3}$ for any s under consideration, hence

$$\mathcal{I}_2 \ll (X + Y)\sqrt{T}(\log T)^2 \mathcal{X}_1 \mathcal{Y}_1.$$

The integral \mathcal{I}_4 can be estimated in a similar way. The relation (9) is proved.

3.1.2. *Proof of (10).* In view of the inequalities

$$|X(\tfrac{1}{2} + it_n(\phi))|^2 \leq \left(\sum_{n \leq X} \frac{|x_n|}{\sqrt{n}}\right)^2 \leq X \sum_{n \leq X} \frac{|x_n|^2}{n}$$

it suffices to consider only the sum over the segment $c < t_n(\phi) \leq T$. Next, we may set $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$. Then we have

$$\begin{aligned} \sum_{c < t_n(\phi) \leq T} |X(\tfrac{1}{2} + it_n(\phi))|^2 &= \sum_{c < t_n(\phi) \leq T} X(\tfrac{1}{2} + it_n(\phi)) X_1(\tfrac{1}{2} - it_n(\phi)) \\ &= \frac{1}{2\pi i} \int_{\square} X(s) X_1(1-s) \frac{\Delta'(s)}{\Delta(s) - e^{2i\phi}} ds, \end{aligned}$$

where \square stands for the rectangular contour defined in Section 3.1.1. Denoting the integrals \mathcal{I}_k , $1 \leq k \leq 4$ as above, we get

$$\begin{aligned} \mathcal{I}_1 &= \frac{1}{2\pi i} \int_{a+ic}^{a+iT} X(s) X_1(1-s) \frac{\Delta'(s)}{\Delta(s) - e^{2i\phi}} ds \\ &= -\frac{1}{2\pi i} \int_{a+ic}^{a+iT} X(s) X_1(1-s) \frac{\Delta'}{\Delta}(s) \sum_{k=1}^{\infty} e^{-2ik\phi} \Delta(s)^k ds. \end{aligned}$$

Estimating the integrand as in Section 3.1.1 we find

$$\mathcal{I}_1 \ll X \mathcal{X}_1^2 \int_c^T \frac{\log t \, dt}{\sqrt{t}} \ll X \sqrt{T} (\log T) \mathcal{X}_1^2 \ll X \sqrt{T} (\log T)^2 \sum_{n \leq X} \frac{|x_n|^2}{n}.$$

Furthermore,

$$\mathcal{I}_3 = -\frac{1}{2\pi} \int_c^T X(1-a+it) X_1(a-it) \frac{\Delta'(1-a+it)}{\Delta(1-a+it) - e^{2i\phi}} dt,$$

and the relations $\overline{X}(s) = X_1(\overline{s})$, $\overline{X}_1(s) = X(\overline{s})$ imply

$$\begin{aligned}\overline{\mathcal{I}}_3 &= -\frac{1}{2\pi i} \int_c^T X_1(1-s)X(s) \frac{\Delta'(1-s)}{\Delta(1-s) - e^{-2i\phi}} ds \\ &= -\frac{1}{2\pi i} \int_{a+ic}^{a+iT} X(s)X_1(1-s) \frac{\Delta'}{\Delta}(s) \frac{ds}{1 - e^{-2i\phi}\Delta(s)} = -j_1 + \mathcal{I}_1,\end{aligned}$$

where

$$j_1 = \frac{1}{2\pi i} \int_{a+ic}^{a+iT} X(s)X_1(1-s) \frac{\Delta'}{\Delta}(s) ds.$$

Lemma 5 with $f(s) \equiv 1$ (that is, $\alpha_1 = 1$, $\alpha_n = 0$ for $n > 1$), $\gamma = 0$ and $Y(s) = X_1(s)$, applied to j_1 yields

$$j_1 = -\frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) \sum_{n \leq X} \frac{|x_n|^2}{n} + O(X(\log T)^3 \mathcal{X}_0^2).$$

Using the above bound for \mathcal{I}_1 , we derive

$$\mathcal{I}_3 = \frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) \sum_{n \leq X} \frac{|x_n|^2}{n} + O\left(X\sqrt{T}(\log T)^2 \sum_{n \leq X} \frac{|x_n|^2}{n}\right) + O(X(\log T)^3 \mathcal{X}_0^2).$$

Estimating \mathcal{I}_2 and taking into account the bounds

$$X(s) \ll X^{1-\sigma} \mathcal{X}_1, \quad X_1(1-s) \ll X^\sigma \mathcal{X}_1, \quad \frac{\Delta'(s)}{\Delta(s) - e^{2i\phi}} \ll \log T$$

for $s = \sigma + iT$, $1-a \leq \sigma \leq a$, $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$, we get

$$\mathcal{I}_2 \ll X(\log T) \mathcal{X}_1^2 \ll X(\log T)^2 \sum_{n \leq X} \frac{|x_n|^2}{n}.$$

The integral \mathcal{I}_4 can be estimated in a similar way. Thus, formula (10) is proved.

3.2. Proof of Theorem 1. Suppose that $k = p/q$, where $p > q \geq 1$, $(p, q) = 1$. Let $\kappa = 1/q$, $r = p - q$, $\xi = T^{1/(4p)}$. We define the coefficients for the polynomials in (6) by

$$X(s) = \left(\sum_{n \leq \xi} \frac{d_\kappa(n)}{n^s} \right)^p = \sum_{n \leq \xi^p} \frac{d_{\kappa p}(n; \xi)}{n^s}, \quad Y(s) = \left(\sum_{n \leq \xi} \frac{d_\kappa(n)}{n^s} \right)^r = \sum_{n \leq \xi^r} \frac{d_{\kappa r}(n; \xi)}{n^s},$$

where $d_{\kappa m}(n; \xi)$ is given by

$$d_{\kappa m}(n; \xi) = \sum_{\substack{n=n_1 \cdots n_m \\ n_1, \dots, n_m \leq \xi}} d_\kappa(n_1) \cdots d_\kappa(n_m).$$

By Lemma 7 it is easy to see that $d_{\kappa m}(n; \xi) = d_{\kappa m}(n)$ for $m \leq \xi$ and $0 \leq d_{\kappa m}(n, \xi) \leq d_{\kappa m}(n)$ for $m > \xi$. Next, we consider the moment (7). By (9),

$$S_1(T) = \frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) (e^{-2i\phi} \Sigma_1 + \Sigma_2) + O(R_1),$$

where

$$\begin{aligned}\Sigma_1 &= \sum_{m \leq \xi^p, mn \leq \xi^r} \frac{d_{\kappa p}(m; \xi) d_{\kappa r}(mn; \xi)}{mn} \leq \sum_{n \leq \xi^r} \frac{d_{\kappa r}(n)}{n} \sum_{l|n} d_{\kappa p}(l) \\ &= \sum_{n \leq \xi^r} \frac{d_{\kappa r}(n) d_{\kappa p+1}(n)}{n} \ll (\log T)^{\kappa r(\kappa p+1)},\end{aligned}$$

$$\begin{aligned}\Sigma_2 &= \sum_{m \leq \xi^r, mn \leq \xi^p} \frac{d_{\kappa r}(m; \xi) d_{\kappa p}(mn; \xi)}{mn} \geq \sum_{n \leq \xi} \frac{d_{\kappa p}(n)}{n} \sum_{l|n} d_{\kappa r}(l) \\ &= \sum_{n \leq \xi} \frac{d_{\kappa p}(n) d_{\kappa r+1}(n)}{n} \gg (\log T)^{\kappa p(\kappa r+1)},\end{aligned}$$

and

$$\begin{aligned}R_1 &\ll (\xi^p + \xi^r) \sqrt{T} (\log T)^2 \sum_{n \leq \xi^p} \frac{d_{\kappa p}(n; \xi)}{n} \sum_{m \leq \xi^r} \frac{d_{\kappa r}(m; \xi)}{m} \\ &\ll T^{3/4} (\log T)^{2+\kappa(p+r)} \ll T^{4/5}.\end{aligned}$$

Thus,

$$|S_1(T)| \geq \frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) (\Sigma_2 - \Sigma_1) + O(T^{4/5}) \gg T (\log T)^{1+\kappa p(\kappa r+1)} \gg T (\log T)^{k^2+1}.$$

On the contrary, using Hölder's inequality we get

$$\begin{aligned}|S_1(T)| &\leq \left(\sum_{0 < t_n(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n(\phi))|^{2k} \right)^{1/(2k)} \times \\ &\quad \times \left(\sum_{0 < t_n(\phi) \leq T} |X(\tfrac{1}{2} + it_n(\phi))|^{2k/(2k-1)} \cdot |Y(\tfrac{1}{2} + it_n(\phi))|^{2k/(2k-1)} \right)^{1-1/(2k)} \\ &= \left(\sum_{0 < t_n(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n(\phi))|^{2k} \right)^{1/(2k)} (S_2(T))^{1-1/(2k)},\end{aligned}$$

where $S_2(T)$ is given by (8). In view of (10) we find that

$$|S_2(T)| = \frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) \sum_{n \leq \xi^p} \frac{d_{\kappa p}^2(n; \xi)}{n} + O(\xi^p \sqrt{T} (\log T)^{k^2+1}) \ll T (\log T)^{k^2+1}.$$

Hence,

$$\sum_{0 < t_n(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n(\phi))|^{2k} \geq \frac{(S_1(T))^{2k}}{(S_2(T))^{2k-1}} \gg T (\log T)^{k^2+1}.$$

Theorem 1 is proved.

3.3. Proof of Corollary 4. Our argument follows Soundararajan [11]. Taking $X = Y$, resp. $x_n = y_n$ in (6), we get

$$S_1(T) = \sum_{0 < t_n(\phi) \leq T} \zeta\left(\frac{1}{2} - it_n(\phi)\right) |X\left(\frac{1}{2} + it_n(\phi)\right)|^2$$

for (7). Comparing with (8) we find

$$(11) \quad |S_1(T)| \leq S_2(T) \max_{0 < t_n \leq T} |\zeta(\frac{1}{2} + it_n(\phi))|.$$

Now let $L = \exp(\sqrt{\log X \log \log X})$ where X is a sufficiently large parameter which will be chosen later. Following Soundararajan [11], we define $x_n = n^{\frac{1}{2}} f(n)$, where f is the multiplicative function such that $f(p^k) = 0$ for all primes p and positive integers $k \geq 2$,

$$f(p) = \frac{L}{\sqrt{p} \log p}$$

for all primes p satisfying $L^2 \leq p \leq \exp((\log L)^2)$, and $f(p) = 0$ for all other primes. We observe that

$$\mathcal{X}_0 = \max_{n \leq X} \sqrt{n} f(n) \leq L^m \prod_{j=1}^m \frac{1}{\log p_j},$$

where p_1, \dots, p_m are the least distinct m prime numbers in $[L^2, \exp((\log L)^2)]$ for which $n = p_1 \dots p_m \leq X$. Since $X \geq n \geq L^{2m}$ then $L^m \leq X^{\frac{1}{2}}$ and $\mathcal{X}_0 < L^m \leq X^{\frac{1}{2}}$. Moreover, since $f(n) \leq 1$ for any n , we find

$$\mathcal{X}_1 = \sum_{n \leq X} \frac{f(n)}{\sqrt{n}} \leq \sum_{n \leq X} \frac{1}{\sqrt{n}} \ll X^{\frac{1}{2}}$$

as well as

$$\begin{aligned} \mathcal{X}_2 &= \sum_{n \leq X} \frac{|x_n|^2}{n} = \sum_{n \leq X} f^2(n) = \sum_{\substack{n=p_1 \dots p_m \leq X \\ L^2 < p_1 \dots p_m \leq e^{L^2}}} \frac{L^{2m}}{(p_1 \log p_1 \dots p_m \log p_m)^2} \\ &\leq \prod_{L^2 < p \leq e^{L^2}} \left(1 + \frac{L^2}{p^2 \log^2 p}\right) < \exp\left(L^2 \sum_{p > L^2} \frac{1}{p^2 \log^2 p}\right) < e. \end{aligned}$$

Inserting these bounds in the asymptotic formulae of Proposition 9 yields

$$S_1(T) = (1 + e^{-2i\phi}) \frac{T}{2\pi} \left(\log \frac{T}{2\pi e}\right) \sum_{mn \leq X} \frac{f(m)f(mn)}{\sqrt{n}} + O(X^2 T^{\frac{1}{2}} (\log T)^2)$$

and

$$S_2(T) = \frac{T}{2\pi} \left(\log \frac{T}{2\pi e}\right) \sum_{n \leq X} |f(n)|^2 + O(X T^{\frac{1}{2}} (\log T)^2 + X^2 (\log T)^3).$$

Let $X = T^{\frac{1}{4}-\epsilon}$, then the main terms in the latter formulae dominate the error terms and we may deduce from (11) that

$$\max_{0 < t_n \leq T} |\zeta(\tfrac{1}{2} + it_n(\phi))| \geq \frac{|S_1(T)|}{S_2(T)} \gg \left(\sum_{mn \leq X} \frac{f(m)f(mn)}{\sqrt{n}} \right) \left(\sum_{n \leq X} |f(n)|^2 \right)^{-1}.$$

Soundararajan [11] proved that the right hand-side is $\geq \exp\left((1+o(1))\sqrt{\frac{\log X}{\log \log X}}\right)$ which gives the desired estimate by letting $\epsilon \rightarrow 0$.

4. THE THIRD MOMENT

Corollary 4 shows that large values of the zeta-function appear on all straight lines through the origin with the imaginary axis as only possible exception. More subtle information on the value-distribution with respect to half-lines can be derived from the third discrete moment. Our first aim is the asymptotic formula of Theorem 2, and we conclude with the proof of Corollary 3.

4.1. Proof of Theorem 2. The method of proof is along the lines of Kalpokas and Steuding [7]. It suffices to evaluate

$$S(T) = \sum_{c < t_n(\phi) \leq T} \zeta(\tfrac{1}{2} + it_n(\phi)) \zeta^2(\tfrac{1}{2} - it_n(\phi)),$$

where $c > 32\pi$ is an absolute constant and $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$ for some ν . Setting $a = 1 + (\log T)^{-1}$, we find by Cauchy's theorem

$$\begin{aligned} S(T) &= \frac{1}{2\pi i} \left\{ \int_{a+ic}^{a+iT} + \int_{a+iT}^{1-a+iT} + \int_{1-a+iT}^{1-a+ic} + \int_{1-a+ic}^{a+ic} \right\} \zeta(s) \zeta(1-s)^2 \frac{\Delta'(s)}{\Delta(s) - e^{2i\phi}} ds \\ &= \sum_{k=1}^4 \mathcal{I}_k, \end{aligned}$$

say.

First we consider \mathcal{I}_1 . In view of (4) we obtain similar to the analogous case of \mathcal{I}_1 in Section 3.1.1

$$\begin{aligned} \mathcal{I}_1 &= -\frac{e^{-2i\phi}}{2\pi i} \int_{a+ic}^{a+iT} \zeta^3(s) \Delta(1-s) \frac{\Delta'}{\Delta}(s) \left(1 + \sum_{k=1}^{\infty} e^{-2ik\phi} \Delta^k(s) \right) ds \\ &= -e^{-2i\phi} (j_1 + e^{-2i\phi} j_2 + j_3), \end{aligned}$$

where

$$\begin{aligned} j_1 &= \frac{1}{2\pi i} \int_{a+ic}^{a+iT} \zeta^3(s) \Delta(1-s) \frac{\Delta'}{\Delta}(s) ds, \\ j_2 &= \frac{1}{2\pi i} \int_{a+ic}^{a+iT} \zeta^3(s) \frac{\Delta'}{\Delta}(s) ds, \\ j_3 &= \frac{1}{2\pi i} \int_{a+ic}^{a+iT} \zeta^3(s) \frac{\Delta'}{\Delta}(s) \sum_{k=1}^{\infty} e^{-2i(k+1)\phi} \Delta^k(s) ds. \end{aligned}$$

By Gonek's Lemma 6 (with $m = 1$) and Lemma 8 we have

$$j_1 = - \sum_{n \leq \frac{T}{2\pi}} d_3(n) \log n + O(T^{\frac{1}{2}+\varepsilon}) = -\frac{T}{2\pi} P_3\left(\log \frac{T}{2\pi}\right) + O(T^{\frac{1}{2}+\varepsilon}),$$

where $P_3(x)$ is a computable polynomial of degree three.

Next, Lemma 5 with $X(s) \equiv 1$, $Y(s) \equiv 1$, applied to j_2 , leads to

$$j_2 = -\frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) + O((\log T)^6).$$

Finally, by standard arguments we obtain

$$j_3 \ll \int_c^T \frac{\log t}{(a-1)^3 \sqrt{t}} dt \ll \sqrt{T} (\log T)^4.$$

Hence,

$$\mathcal{I}_1 = e^{-2i\phi} \frac{T}{2\pi} P_3\left(\log \frac{T}{2\pi}\right) + e^{-4i\phi} \frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) + O(T^{\frac{1}{2}+\varepsilon}).$$

Further, transforming the integral \mathcal{I}_3 we find

$$\begin{aligned} \overline{\mathcal{I}_3} &= -\frac{1}{2\pi i} \int_{a+ic}^{a+iT} \zeta(1-s) \zeta^2(s) \frac{\Delta'(1-s) ds}{\Delta(1-s) - e^{-2i\phi}} \\ &= -\frac{1}{2\pi i} \int_{a+ic}^{a+iT} \zeta^3(s) \Delta(1-s) \frac{\Delta'}{\Delta}(s) \sum_{k=0}^{\infty} e^{-2i\phi} \Delta(s)^k ds. \end{aligned}$$

The latter expression equals $e^{2i\phi} \mathcal{I}_1$, hence we may deduce (do not forget to conjugate)

$$\mathcal{I}_3 = \frac{T}{2\pi} P_3\left(\log \frac{T}{2\pi}\right) + e^{2i\phi} \frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) + O(T^{\frac{1}{2}+\varepsilon}).$$

In order to estimate the integral \mathcal{I}_2 over the top and bottom edges we write

$$\begin{aligned} F(s) &= \zeta^3(s) \Delta(1-s) \frac{\Delta'}{\Delta}(s) \frac{1}{\Delta(s) - e^{2i\phi}} \\ &= \zeta^3(1-s) \Delta(s) \frac{\Delta'}{\Delta}(s) \left(1 + \frac{e^{2i\phi}}{\Delta(s) - e^{2i\phi}} \right). \end{aligned}$$

Since $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$ for some ν , the inequality $|\Delta(s) - e^{2i\phi}| > \frac{1}{3}$ from Paragraph 3.1.1 holds over the segment of integration. Using the bound $|\zeta(\sigma + it)| \ll t^{(1-\sigma)/3} \log t$, for $s = \sigma + iT$, $\frac{1}{2} \leq \sigma \leq a$ (see [6]), we get

$$|F(s)| \ll (\log T) |\zeta^3(s) \Delta(1-s)| \ll (\log T) (T^{\frac{1}{3}(1-\sigma)})^3 T^{\sigma-\frac{1}{2}} \ll \sqrt{T} (\log T)^4.$$

In the case $1-a \leq \sigma \leq \frac{1}{2}$ we have

$$|F(s)| \ll (\log T) |\zeta^3(1-s) \Delta(s)| \ll (\log T) (T^{\frac{1}{3}(1-(1-\sigma))} \log T)^3 T^{\frac{1}{2}-\sigma} \ll \sqrt{T} (\log T)^4.$$

Thus, $\mathcal{I}_2 \ll \sqrt{T} (\log T)^4$. Finally, the bound $\mathcal{I}_4 = O(1)$ is obvious. Collecting together the above results, we obtain

$$S(T) = (1 + e^{-2i\phi}) \frac{T}{2\pi} P_3 \left(\log \frac{T}{2\pi} \right) + (e^{2i\phi} + e^{-4i\phi}) \frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) + O(T^{\frac{1}{2}+\varepsilon})$$

Now it remains to note that we must multiply $S(T)$ by $e^{4i\phi}$ to obtain

$$\sum_{0 < t_n(\phi) \leq T} \zeta^3\left(\frac{1}{2} + it_n(\phi)\right) = e^{4i\phi} S(T) + O(1).$$

The theorem is proved.

REMARK. It is possible to compute the coefficients of the polynomial P_3 as follows. Define the polynomial $P_2(u) = A_2 u^2 + A_1 u + A_0$ by the relation

$$\sum_{n \leq x} d_3(n) = x P_2(\log x) + o(x),$$

which is a special case of the asymptotics from Lemma 8. We get

$$P_2(\log x) = \operatorname{res}_{s=1} \left(\frac{x^s \zeta^3(s)}{s} \right)$$

and hence $A_2 = \frac{1}{2}$, $A_1 = 3\gamma - 1$, $A_0 = 1 + 3(\gamma^2 - \gamma + \gamma_1)$, where γ, γ_1, \dots are the coefficients of Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma + \gamma_1(s-1) + \dots$$

Thus, using the definition of $P_3(u)$ and Abel's summation formula, we find

$$P_3(u) = u P_2(u) - P_2(u) + P_2'(u) - P_2''(u) = \sum_{k=0}^3 B_k u^k,$$

where $B_3 = A_2 = \frac{1}{2}$, $B_2 = A_1 - A_2 = 3\gamma - \frac{3}{2}$, $B_1 = A_0 - A_1 + 2A_2 = 3(\gamma_1 + (1-\gamma)^2)$ and $B_0 = -B_1 = -3(\gamma_1 + (1-\gamma)^2)$. For the values of the coefficients γ_j and P_2 we refer to [12].

4.2. Proof of Corollary 3. First we observe for any non-negative integer ℓ

$$\begin{aligned} & \sum_{t_n^\pm(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n^\pm(\phi))|^{2\ell+1} \\ &= \frac{1}{2} \sum_{t_n(\phi) \leq T} \left(|\zeta(\tfrac{1}{2} + it_n(\phi))|^{2\ell+1} \pm e^{-(2\ell+1)i\phi} \zeta(\tfrac{1}{2} + it_n(\phi))^{2\ell+1} \right) \end{aligned}$$

(with the same choice of signs on either side of the equation). In view of Theorem 2 and Theorem 1 with $k = \frac{3}{2}$ we get

$$\sum_{t_n^\pm(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n^\pm(\phi))|^3 \gg T(\log T)^{\frac{13}{4}}.$$

Since the number of intersection points $t_n(\phi) \leq T$ is bounded by $T \log T$ (see Theorem 1 from [7]) and

$$\sum_{0 < t_n^\phi \leq T} |\zeta(\tfrac{1}{2} + it_n(\phi))|^2 \ll T(\log T)^2$$

(see Theorem 2 from [7]), we find

$$\begin{aligned} \sum_{t_n^\pm(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n^\pm(\phi))|^3 &\ll \max_{t_n^\pm(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n^\pm(\phi))| \sum_{0 < t_n^\phi \leq T} |\zeta(\tfrac{1}{2} + it_n(\phi))|^2 \\ &\ll \max_{t_n^\pm(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n^\pm(\phi))| T(\log T)^2. \end{aligned}$$

Comparing both estimates we arrive at

$$\max_{t_n^\pm(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n^\pm(\phi))| \gg (\log T)^{\frac{5}{4}},$$

If we assume Riemann Hypothesis we can use the following estimate (see Theorem 1 from [1]) that holds for any non-negative real k , uniformly for $\phi \in [0, \pi)$, as $T \rightarrow \infty$,

$$\sum_{0 < t_n^\phi \leq T} |\zeta(\tfrac{1}{2} + it_n(\phi))|^{2k} \ll T(\log T)^{k^2+1+\epsilon}.$$

We have

$$\begin{aligned} T(\log T)^{\frac{13}{4}} &\ll \sum_{t_n^\pm(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n^\pm(\phi))|^3 \ll \max_{t_n^\pm(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n^\pm(\phi))|^\alpha \sum_{0 < t_n^\phi \leq T} |\zeta(\tfrac{1}{2} + it_n(\phi))|^{3-\alpha} \\ &\ll \max_{t_n^\pm(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n^\pm(\phi))|^\alpha T(\log T)^{(\frac{3}{2}-\frac{\alpha}{2})^2+1+\epsilon}. \end{aligned}$$

Comparing both sides we arrive at

$$\max_{t_n^\pm(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n^\pm(\phi))| \gg (\log T)^{\frac{3}{2}-\frac{\alpha}{4}-\frac{\epsilon}{\alpha}}.$$

After choosing $\alpha = 2\delta$ and $\epsilon = \delta^2$, where δ is positive and arbitrary small we get

$$\max_{t_n^\pm(\phi) \leq T} |\zeta(\tfrac{1}{2} + it_n^\pm(\phi))| \gg (\log T)^{\frac{3}{2}-\delta}.$$

which proves the corollary.

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